Course Notes for Signals and Systems

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Contents

Chapter 1

Math Review

1.1 Trigonometric Identities

It will be useful to memorize $\sin\theta, \cos\theta, \tan\theta$ values for $\theta = 0, \pi/3, \pi/4, \pi/2$ and $\pi \pm \theta$, $2\pi - \theta$ for the above values of θ. The values of $\sin\theta$ and $\cos\theta$ for these values are given below

Table 1.1: Some sine and cosine values to memorize

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	1	0
$\pi/6 = 30°$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\pi/4 = 45°$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\pi/3 = 60°$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\pi/2 = 90°$	1	0	∞

The following identities involving sine and cosine functions will be useful

$$\begin{aligned}
\sin(\theta \pm \phi) &= \sin\theta\ \cos\phi \pm \cos\theta\ \sin\phi \\
\cos(\theta \pm \phi) &= \cos\theta\ \cos\phi \mp \sin\theta\ \sin\phi \\
\sin\theta \sin\phi &= \frac{1}{2}\left[\cos(\theta - \phi) - \cos(\theta + \phi)\right] \\
\cos\theta \cos\phi &= \frac{1}{2}\left[\cos(\theta - \phi) + \cos(\theta + \phi)\right] \\
\sin\theta \cos\phi &= \frac{1}{2}\left[\sin(\theta - \phi) + \sin(\theta + \phi)\right]
\end{aligned} \tag{1.1}$$

The following special case of the above formulas are also very useful to commit to memory

$$
\begin{aligned}
\sin(\pi/2 \pm \phi) &= \cos\phi \\
\cos(\pi/2 \pm \phi) &= \mp \sin\phi \\
\sin(\pi \pm \phi) &= \mp \sin\phi \\
\cos(\pi \pm \phi) &= -\cos\phi \\
\cos^2\theta &= \frac{1}{2}\left(1+\cos 2\theta\right) \\
\sin^2\theta &= \frac{1}{2}\left(1-\cos 2\theta\right) \\
\cos(2\theta) &= 2\cos^2\theta - 1 = 1 - 2\sin^2\theta
\end{aligned}
$$

1.2 Magnitude and Angle Representation

Any two real numbers a and b can be written as $a = r\cos\theta$ and $b = r\sin\theta$, where $r \geq 0$ and $0 \leq \theta < 2\pi$. Such and r and θ are given by

$$r = \sqrt{a^2 + b^2}, \tag{1.2}$$

$$\theta = \tan^{-1}\left(\frac{b}{a}\right). \tag{1.3}$$

There can be an ambiguity of π in the result for θ and this should be resolved based on the sign of b.

1.3 Complex Numbers

We will use the letter j to refer to the imaginary number $\sqrt{-1}$. Even though j is not a real number, we can perform all arithmetic operations such as addition, subtraction, multiplication, division with j using the algebra of real numbers.

1.3.1 Cartesian Form

A complex number z is any number of the form $z = x + jy$, where x is called the real part of z and y is called the imaginary part of z. They are denoted by $\Re\{z\}$ and $\Im\{z\}$, respectively. Note: The imaginary part is not jy, rather it is only y. It is important to stick to this terminology, otherwise computations can go wrong. Often, it is useful to think of a complex number $z = x + jy$ as a vector in a two-dimensional plane as shown in the figure below, where x is the X-coordinate and y is Y-coordinate of the vector. Due to this relationship between a complex number and the corresponding vector, we will abuse the terminology and use the terms complex number and vector interchangeably, if the context should resolve any possible ambiguity. For example, a complex number is said to lie in the first quadrant (or, second quadrant etc) if the corresponding vector lies in the first quadrant (or, second quadrant etc).

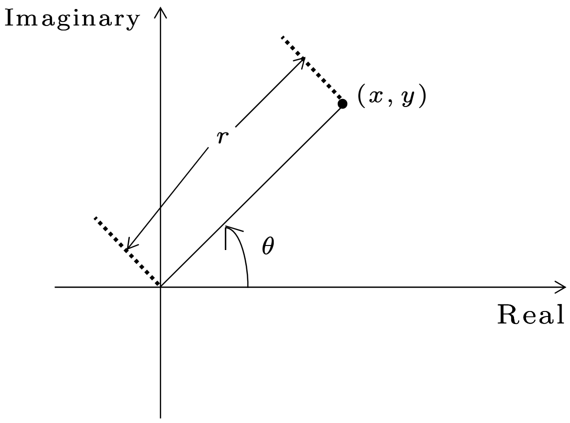

Figure 1.1: Argand diagram showing a complex number as a point/vector in the 2-D complex plane

1.3.2 Magnitude and Phase

When a complex number is thought of as a vector in two dimensions, the X coordinate x and the Y coordinate y can be expressed in terms of the length of the vector r and the angle made by this vector with the positive X-axis, namely θ. Since $x = r\cos\theta$ and $y = r\sin\theta$, z can be expressed as

$$z = r\cos\theta + jr\sin\theta, \tag{1.4}$$

where θ can be in degrees or radians (usually radians) and recall that 2π rad $= 360\,°$. r is called the magnitude of z, denoted by $|z|$ and θ is called the phase of the complex number z, denoted by $\arg\{z\}$ or $\angle z$.

It is easy to see that x, y, r and θ are related according to

$$x = \Re\{z\} = r\ \cos\theta, \qquad y = \Im\{z\} = r\ \sin\theta$$
$$r = |z| = \sqrt{x^2 + y^2}, \qquad \theta = \angle z = \tan^{-1}\left(\frac{y}{x}\right) \tag{1.5}$$

⚠Note: The magnitude r represents the *length* of the vector and hence, has to be positive. It does not make sense for the magnitude of a complex number to be negative.

1.3.3 Euler's Formula

The cosine, sine and exponential functions have infinite series (Maclaurin's series) expansions given by

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} - \dots \tag{1.6}$$
$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \frac{\theta^9}{9!} - \dots \tag{1.7}$$
$$e^\theta = 1 + \frac{\theta}{1!} + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{\theta^5}{5!} + \dots \tag{1.8}$$

where θ is in radians.

By replacing θ by $j\theta$ and $-j\theta$, respectively, in (1.8), we get the following two equations.

$$e^{j\theta} = 1 + \frac{j\theta}{1!} + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + \frac{(j\theta)^5}{5!} + \dots \tag{1.9}$$
$$e^{-j\theta} = 1 - \frac{j\theta}{1!} + \frac{(j\theta)^2}{2!} - \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} - \frac{(j\theta)^5}{5!} + \dots \tag{1.10}$$

From (1.9), (1.10), (1.6) and (1.7) the following relationship can be seen to be true. Equation 1.11 is called Euler's formula. We will repeatedly use this identity in the notes and, hence, you should memorize and develop a familiarity with these four formulae.

$$e^{j\theta} = \cos\theta + j\sin\theta \tag{1.11}$$
$$e^{-j\theta} = \cos\theta - j\sin\theta$$
$$\cos\theta = \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right)$$
$$\sin\theta = \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right)$$

When $\theta = \pi$, (1.11) reduces to $e^{j\pi} = -1$ or, equivalently

$$\boxed{e^{j\pi} + 1 = 0},$$

which is known as Euler's identity. The famous physicist Richard Feynman called this "our jewel" and "the most remarkable formula in mathematics" `https://en.wikipedia.org/wiki/Mathematical_beauty`.

1.3.4 Polar form or Exponential form

We already saw that a complex $z = x + jy$ can be written as $z = r\cos\theta + jr\sin\theta$. Using Euler's identities z can be written as

$$z = x + jy = r\cos\theta + jr\sin\theta = re^{j\theta}. \tag{1.12}$$

$re^{j\theta}$ is known as the polar form or exponential form.

In summary, the same complex number z can be written in Cartesian form as $x + jy$ or in polar form as $re^{j\theta}$ and it is very important to be able to convert a complex number from cartesian form to exponential form and vice versa.

Example 1.3.1. *It is very useful to know the polar form for often used complex numbers such as* $1, j, -j, -1$. *They are given by*

$$\boxed{1 = e^{j0}, -1 = e^{j\pi}, j = e^{j\frac{\pi}{2}}, -j = e^{-j\frac{\pi}{2}}}.$$

Example 1.3.2. *Let* $z = 2e^{j\frac{\pi}{4}}$. *Find the magnitude and angle of* z. *The answer is* $|z| = 2$ *and* $\theta = \pi/4$. *Many students will first convert* z *to Cartesian form, write it as* $z = \sqrt{2}+j\sqrt{2}$ *and then compute* $|z|$ *as* $\sqrt{(\sqrt{2})^2 + (\sqrt{2})^2}$. *While this is not wrong, this is time consuming and misses the point. You should train yourself to see that* z *is already given in polar form, i.e., as* $re^{j\theta}$ *with* $r = 2$ *and* $\theta = \pi/4$. *Since* r *represents the magnitude of* z, *you can directly read off the magnitude as 2 and angle as* $\pi/4$. *In my experience, students have difficulty with this sometimes well into the course.*

Example 1.3.3. *Let* $z = -2e^{j\frac{\pi}{4}}$. *Find* $|z|$ *and* $\angle z$.

The point of this example is to emphasize that the magnitude of a complex number r *must be positive. It would be wrong to say that* $|z| = -2$. *We must rewrite* z *as* $z = 2e^{j\pi}e^{j\frac{\pi}{4}} = 2e^{j\frac{5\pi}{4}}$ *and interpret* r *as* 2 *instead of* -2 *and the angle as* $\pi + \pi/4 = 5\pi/4$.

Caution: The expression for θ in (1.5) does not identify θ uniquely, since $\tan(\theta) = \frac{y}{x}$ also implies that $\tan(\theta \pm \pi) = \frac{y}{x}$. It is best think of the vector (x, y) and determine which quadrant this vector lies in based on the signs of x, y and then make sure θ corresponds to an angle in the correct quadrant.

Example 1.3.4. *Suppose* $z_1 = \frac{\sqrt{3}}{2} + j\frac{1}{2}$ *and* $z_2 = -\frac{\sqrt{3}}{2} - j\frac{1}{2}$. *It is easy to see that* $\tan^{-1}\left(\frac{y_1}{x_1}\right) = \tan^{-1}\left(\frac{y_2}{x_2}\right)$. *However,* z_1 *is complex number in the first quadrant, whereas* z_2 *is a complex number in the 3rd quadrant. Therefore,* θ_1 *should be* $\pi/6$ *and* θ_2 *should be* $7\pi/6$.

One important aspect of the polar form for a complex number is that adding 2π to the angle does not change the complex number. Particularly,

$$\boxed{re^{j\theta} = re^{j(\theta+2k\pi)} \quad \text{for any integer } k}$$

This fact will be repeatedly used in the course. An immediate example of where this is useful is given in Section 1.3.7.

Example 1.3.5. *Express* $e^{j2\pi}, e^{-j\pi}, e^{j\frac{3\pi}{2}}, e^{j\frac{9\pi}{2}}, e^{j\pi}$ *in Cartesian form.*

1.3.5 Conjugate

The conjugate of a complex number $z = x + jy$ is given by $z^* = x - jy$. When z is written in polar form as $z = re^{j\theta}$, the complex conjugate is given by $z^* = re^{-j\theta}$. In general, to compute the conjugate of a complex number, replace j by $-j$ everywhere.

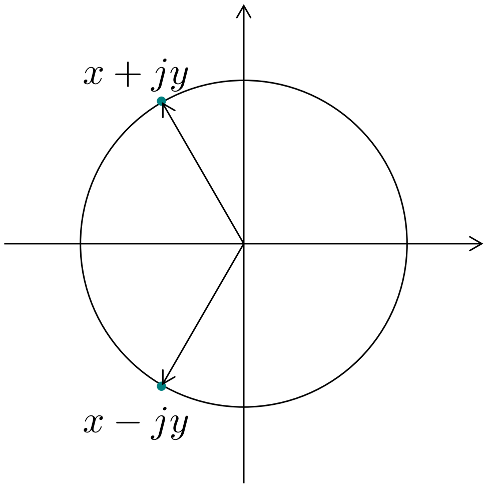

Figure 1.2: Conjugate of a complex number

1.3.6 Operations on two complex numbers

Let $z = x + jy = re^{j\theta}$, $z_1 = x_1 + jy_1 = r_1 e^{j\theta_1}$ and $z_2 = x_2 + jy_2 = r_2 e^{j\theta_2}$. The following facts about addition, subtraction, multiplication and division of complex numbers can be easily verified.

$$z_1 \pm z_2 = (x_1 \pm x_2) + j(y_1 \pm y_2) \tag{1.13}$$

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + j(x_1 y_2 + x_2 y_1) = r_1 r_2 e^{j(\theta_1 + \theta_2)} \tag{1.14}$$

$$|z| = \sqrt{zz^*} = r$$

$$\frac{z_1}{z_2} = \frac{(x_1 + jy_1)}{(x_2 + jy_2)} = \frac{(x_1 + jy_1)(x_2 - jy_2)}{x_2^2 + y_2^2} = \frac{r_1}{r_2} e^{j(\theta_1 - \theta_2)}$$

Typically, adding and subtracting complex numbers will be easier when the Cartesian form is used whereas, multiplication and division will be easier when the polar form is used. It will be important to be able to use both these representations in order to make computations easier.

Based on the above operations, the following facts about complex number can be verified.

$$
\begin{aligned}
(z_1 + z_2)^* &= z_1^* + z_2^* \\
(z_1\ z_2)^* &= z_1^*\ z_2^* \\
\left(\frac{z_1}{z_2}\right)^* &= \frac{z_1^*}{z_2^*} \\
|z_1 - z_2| &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\
|z_1 z_2| &= |z_1||z_2| = r_1 r_2 \\
\left|\frac{z_1}{z_2}\right| &= \frac{|z_1|}{|z_2|} = \frac{r_1}{r_2} \\
zz^* &= x^2 + y^2 = r^2 \\
z + z^* &= 2\Re\{z\} \\
z - z^* &= 2j\Im\{z\}
\end{aligned}
$$

1.3.7 n^{th} power and n^{th} roots of a complex number

Let $z_0 = x_0 + jy_0 = r_0 e^{j\theta_0}$. For any integer n, the nth power of z, z^n is simply obtained by using (1.14) n times. In the polar form, $z_0^n = r_0^n e^{jn\theta_0}$. Just like how the two real numbers 1 and -1 have the same square, different complex numbers can have the same nth power.

Consider the set of distinct complex numbers $z_k = e^{j\theta_0 + \frac{2\pi k}{n}}$. All the z_ks are different have the same nth power for $k = 0, 1, 2, \ldots, n-1$. We can see this by raising z_k to the nth power to get

$$z_k^n = \left(e^{j\theta_0 + \frac{2\pi k}{n}}\right)^n = e^{jn\theta_0 + 2\pi k} = e^{jn\theta_0}. \tag{1.15}$$

The nth root of z_0 is a bit more interesting and tricky. Any complex number z which is the solution to the nth degree equation

$$z^n - z_0 = 0$$

is an nth root of z_0. The fundamental theorem of algebra states that an nth degree equation has exactly n (possibly complex and possibly repeated) roots. Hence, every complex number z_0 has exactly n, nth roots. These roots can be found by using the fact $e^{j\theta} = e^{j(\theta + 2\pi k)}$.

$$
\begin{aligned}
z^n = z_0 \quad &\Rightarrow \quad r^n e^{jn\theta} = r_0 e^{j\theta_0} = r_0 e^{j(\theta_0 + 2k\pi)} \\
&\Rightarrow \quad r = \sqrt[n]{r_0}, \quad , \theta = \frac{\theta_0 + 2k\pi}{n} \text{ for } k = 0, 1, 2, \ldots n-1
\end{aligned} \tag{1.16}
$$

Clearly, computing nth roots is much easier in the polar form than in the cartesian form.

Example 1.3.6. *Find the third roots of unity $\sqrt[3]{1}$*

Since $1 = 1e^{j0}$, this corresponds to $r_0 = 1, \theta_0 = 0$. Hence, the three roots of unity are given by

$$r = 1, \quad \theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}.$$

In cartesian coordinates, they are $(1 + j0)$, $\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)$, $\left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right)$. These are referred to as $1, \omega, \omega^2$ sometimes. The three roots are shown in Figure 1.3.

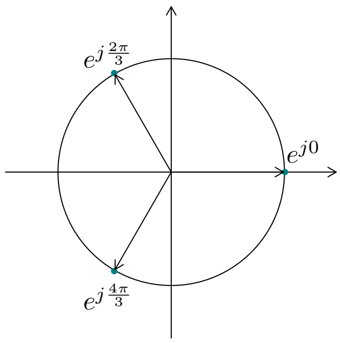

Figure 1.3: Cube roots of unity

1.3.8 Functions of a complex variable

Let $f(z)$ be a complex function of a complex variable z, i.e., for every z, $f(z)$ is a complex number. Note that a real number is also considered as a complex number and, hence, $f(z)$ could have a zero imaginary part. Examples of functions include $f(z) = |z|, f(z) = \arg(z), f(z) = z^n, f(z) = \exp(z)$, etc. The exponential functions can be interpreted using Euler's identity as follows.

$$f(z) = \exp(z) = e^x e^{jy} = e^x \cos y + je^x \sin y$$

The logarithm of a complex number $\ln z$ can be also interpreted using Euler's identity as

$$\ln(z) = \ln\left(re^{j(\theta+2k\pi)}\right) = \ln r + j(\theta + 2k\pi).$$

It can be seen from the above expression that $\ln z$ is not a function of z. However, if we set $k = 0$ in the above expression, then we get what is called the principal value of $\ln z$, denoted by Ln z, which is a function.

Similarly, it is important to realize that for any integer n, the nth power of a complex number is a function of the complex number, i.e., for every complex number z, there is only one complex number z^n. However, for an integer n, the nth root of a complex number is not uniquely defined and hence, is not a function. Often, one may take the root corresponding to $k = 0$ in (1.16) as the default root and hence the principal value. Then, the principal value becomes a function. This is similar to square roots of positive real numbers being defined as the positive numbers. There are interesting examples where careless use of just the principal value as the nth root can lead to fallacious arguments.

1.3.9 Complex functions of a real variable

You may be used to dealing with functions of a variable such as $y = f(x)$, where x is called the independent variable and y is called the dependent variable and typically, y takes real values when x takes real values. In this course, we will be interested in complex functions of a real variable. Often the real variable will represent time or frequency. Such a function, normally denoted as $x(t)$ or $X(\omega)$ is a function which takes a complex value for every real value of the independent variable t or ω. Pay attention to the notation carefully - t or ω now becomes the independent variable and $x(t)$ or $X(\omega)$ now becomes the dependent variable.

We can also think of the complex function as the combination of two real functions of the independent variable, one for the real part of $x(t)$ and one for the imaginary part of $x(t)$.

When dealing with real functions of a real variable, you may be used to plotting the function $x(t)$ as a function of t. However, when $x(t)$ is a complex function, there is a problem in plotting this function since for every value of t, we need to plot a complex number. In this case, we do one of two things - either we plot the real part of $x(t)$ versus t and plot the imaginary part of $x(t)$ versus t, or we plot $|x(t)|$ versus t and $\arg(x(t))$ versus t. Either of these is fine, but we do need two plots to effectively understand how $x(t)$ changes with t.

Example 1.3.7. *Consider the function $x(t) = e^{j2\pi t} = \cos 2\pi t + j \sin 2\pi t$ for all real values of t. This is clearly a complex function of a real variable t. $\Re\{x(t)\}, \Im\{x(t)\}, |x(t)|, arg(x(t))$ are all real functions of the real variable t. Hence, we can plot $\Re\{x(t)\}$ versus t and $\Im\{x(t)\}$ versus t or we can plot $|x(t)|$ versus t and $\angle x(t)$ versus t as shown in Fig. 1.4*

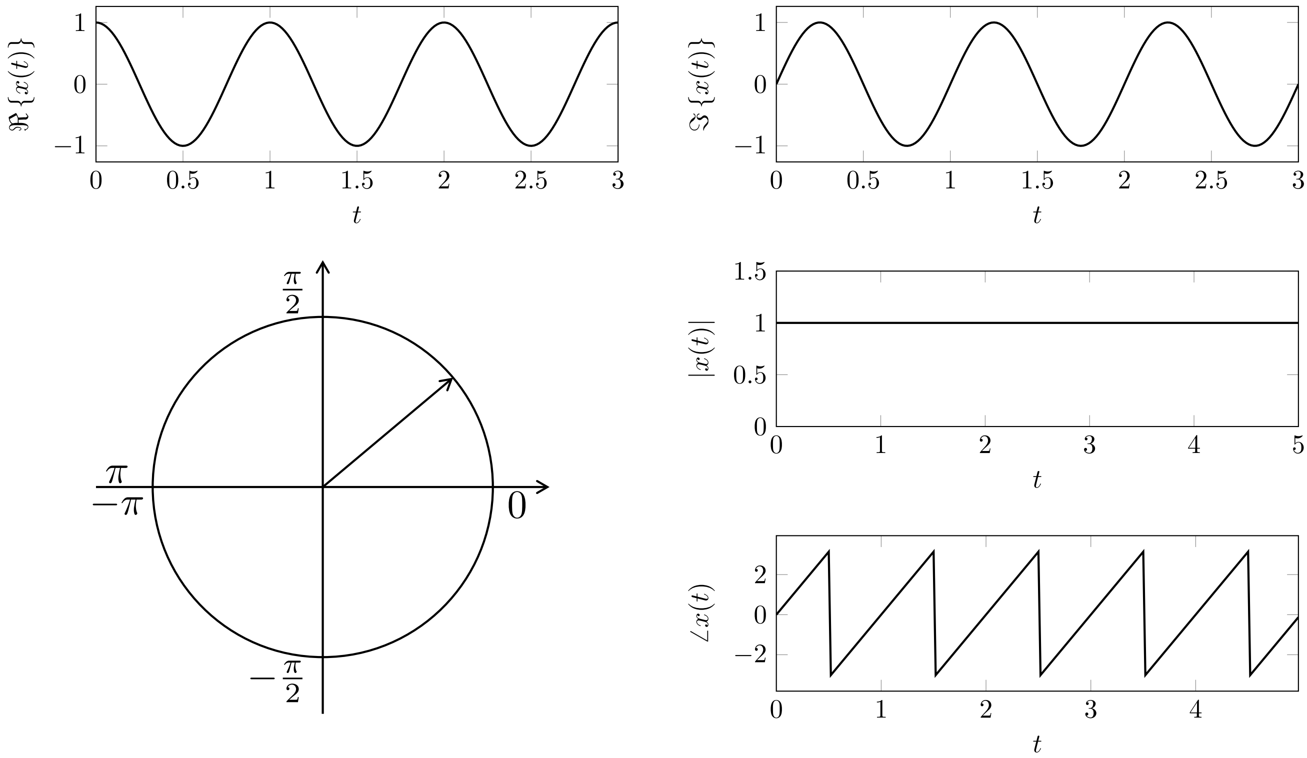

Figure 1.4: Plot of $\Re\{x(t)\}, \Im\{x(t)\}, |x(t)|, \angle x(t)$ versus t for $x(t) = e^{j2\pi t}$.

Example 1.3.8. *Consider the function $x(t) = e^{st}$ where $s = \sigma + j\omega$ is some complex number. This is a complex function of a real variable t. The real part and imaginary part of $x(t)$ are each real functions of a real variable t and can be obtained as follows. Notice that $x(t)$ can be written as*

$$\begin{aligned} x(t) &= e^{\sigma + j\omega}t = e^{\sigma t} e^{j\omega t} \\ &= e^{\sigma t} \left(\cos(\omega t) + j \sin(\omega t) \right) \\ &= \underbrace{e^{\sigma t} \cos(\omega t)}_{\textit{Real part}} + j \underbrace{e^{\sigma t} \sin(\omega t)}_{\textit{imaginary part}} \end{aligned}$$

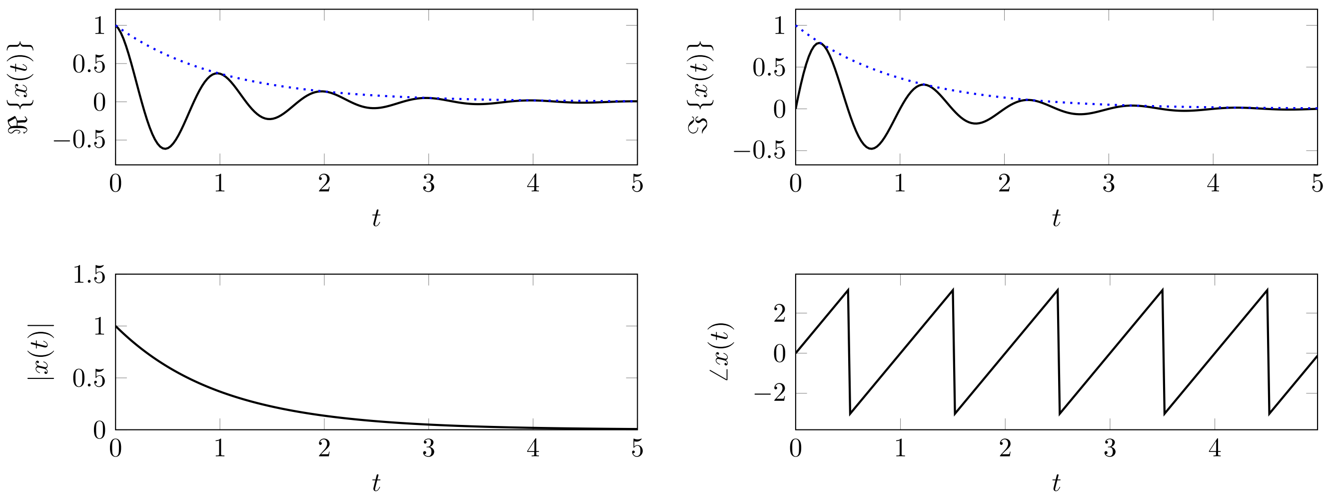

Figure 1.5: Plots of $\Re x(t)$ and $\Im x(t)$ versus t and $|x(t)|$ and $\angle x(t)$ versus t.

The magnitude of $x(t)$ and phase of $x(t)$ are both functions of t given by

$$\begin{aligned} |x(t)| &= e^{\sigma t} \\ \angle x(t) &= \omega t \end{aligned}$$

These are plotted in Fig. 1.5 for $\sigma = -1$ and $\omega = 2\pi$. The dotted blue lines in the figures show $|x(t)|$ and it can be seen that it defines the envelope of the real and imaginary parts of $x(t)$.

It is useful to get insight into what happens to $x(t) = e^{st}$ as $t \to \infty$. It can be seen that if $\sigma < 0$, $x(t) \to 0$ as $t \to \infty$ and if $\sigma > 0$, $x(t)$ becomes undefined at $t \to \infty$ with the $|x(t)| \to \infty$ as $t \to \infty$. Thus, the real part of s determines whether the signal $x(t)$ is bounded or unbounded as $t \to \infty$.

Example 1.3.9. *Consider the function $H(\omega) = \frac{1}{1+j\omega}$, where ω is a real variable. Roughly sketch the magnitude and phase of $H(\omega)$ as a function of ω.*

$$\begin{aligned} H(\omega) &= \frac{1}{1+j\omega} \\ |H(\omega)| &= \frac{1}{\sqrt{1+\omega^2}} \\ \angle(H(\omega)) &= 0 - \tan^{-1}\omega \end{aligned}$$

A plot of $|H(\omega)|$ versus ω and $\angle(H(\omega))$ versus ω is shown in Fig. 1.6.

Example 1.3.10. *Consider the function $X(\omega) = \frac{j\omega}{1+j\omega}$, where ω is a real variable. Roughly sketch the magnitude and phase of $X(\omega)$ as a function of ω.*

$$\begin{aligned} X(\omega) &= \frac{j\omega}{1+j\omega} \\ |X(\omega)| &= \frac{|\omega|}{\sqrt{1+\omega^2}} \\ \angle(X(\omega)) &= \begin{cases} -\frac{\pi}{2} - \tan^{-1}\omega & ,\omega < 0 \\ \frac{\pi}{2} - \tan^{-1}\omega & ,\omega > 0 \end{cases} \end{aligned}$$

A plot of $|X(\omega)|$ versus ω and $\angle(X(\omega))$ versus ω is shown in Fig. 1.7

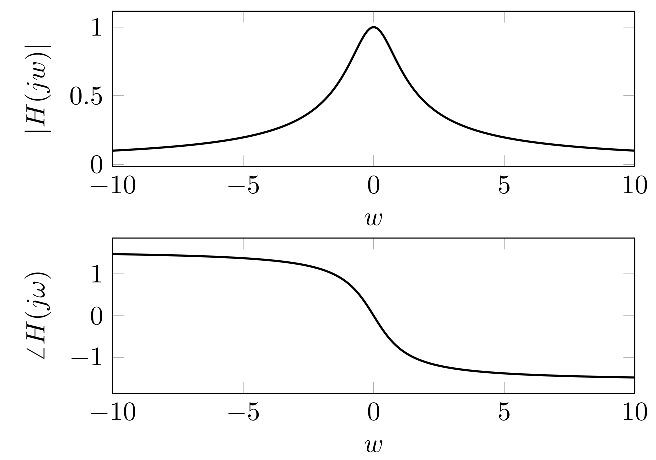

Figure 1.6: Plot of $H(\omega)$ vs ω and $\angle H(\omega)$ versus ω for $H(\omega) = \frac{1}{1+j\omega}$.

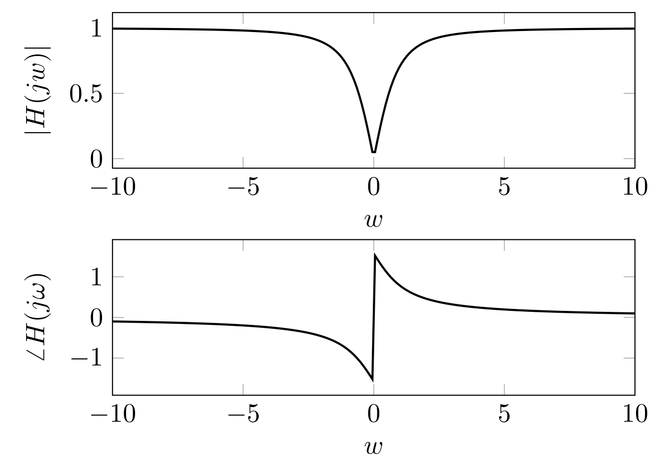

Figure 1.7: Plot of $X(\omega)$ vs ω and $\angle X(\omega)$ versus ω for $X(\omega) = \frac{j\omega}{1+j\omega}$.

1.3.10 Plotting the magnitude and phase of $H(\omega) = e^{ja_1\omega} + e^{ja_2\omega}$ vs ω

One of the tricks that is useful to get some insight into plotting the magnitude and phase of functions of the form $H(j\omega) = e^{ja_1\omega} + e^{ja_2\omega}$ vs ω is to express $e^{ja_1\omega} + e^{ja_2\omega}$ as follows

$$\begin{aligned} e^{ja_1\omega} + e^{ja_2\omega} &= e^{j\left(\frac{a_1+a_2}{2}\omega\right)} \left[e^{j\left(\frac{a_1-a_2}{2}\right)\omega} + e^{-j\left(\frac{a_1-a_2}{2}\right)\omega}\right] \quad (1.17) \\ &= e^{j\left(\frac{a_1+a_2}{2}\omega\right)} 2\cos\left[\left(\frac{a_1-a_2}{2}\right)\omega\right] \end{aligned}$$

Now, it is easy to see that $|H(\omega)| = 2\left|e^{j\left(\frac{a_1+a_2}{2}\omega\right)}\right| \cdot \left|\cos\left[\left(\frac{a_1-a_2}{2}\right)\omega\right]\right|$ which is simply $2\left|\cos\left[\left(\frac{a_1-a_2}{2}\right)\omega\right]\right|$.

Example 1.3.11. *Compute the magnitude and phase of $H(\omega) = e^{j\omega} + e^{j3\omega}$ and determine the values of ω for which $H(\omega) = 0$.*

$$\begin{aligned} H(\omega) = e^{j\omega} + e^{j3\omega} &= e^{j2\omega}\left(e^{-j\omega} + e^{j\omega}\right) \quad (1.18) \\ |H(\omega)| &= 2\cos(\omega) \quad (1.19) \\ \angle H(\omega) &= 2\omega \; sign(\cos\omega) \quad (1.20) \end{aligned}$$

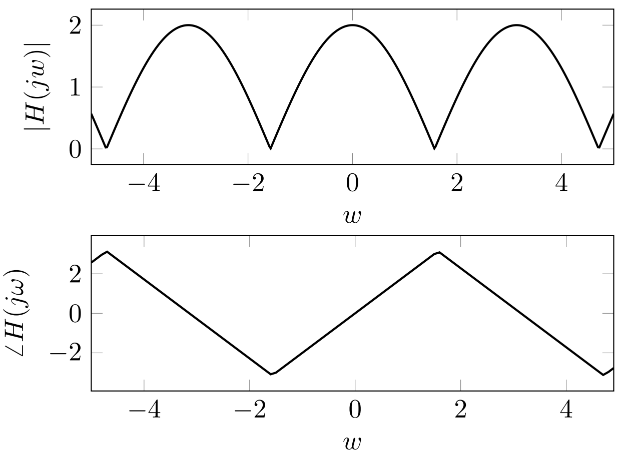

Figure 1.8: Plot of $|H(j\omega)|$ vs ω and $\angle H(j\omega)$ versus ω for $H(j\omega) = H(j\omega) = e^{j\omega} + e^{j3\omega}$.

The advantage of this way to compute the magnitude lies in making it easy to compute values of ω for which $H(\omega) = 0$. These are the values of ω for which $|H(\omega)| = 0$ which are simply given by the values of ω for which $2\cos(\omega) = 0$. These are given by odd multiples of $\pi/2$, i.e., $\omega = (2i+1)\pi/2$ for any integer i.

1.4 Practice Problems

1. Express these numbers in Cartesian form

   a) $2e^{j\pi/4}$

   b) $(1+j)(2-3j)$

   c) $e^{j\pi/4}+e^{j3\pi/4}$

   d) $\frac{2}{e^j}$

   e) $e^{j\pi/3}+e^{-j\pi/3}$

2. Express these numbers in polar form

   a) $1+j$,

   b) $\frac{1}{2}+j\frac{\sqrt{3}}{2}$, $\frac{1}{2}-j\frac{\sqrt{3}}{2}$, $-\frac{1}{2}+j\frac{\sqrt{3}}{2}$, $-\frac{1}{2}-j\frac{\sqrt{3}}{2}$

   c) $(1-j)\left(\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)$

   d) $\frac{1+j}{2j}$

   e) $e^{j\pi/4}+e^{j3\pi/4}$

   f) $1+e^j$

3. Compute the magnitude and phase of the following complex numbers. Do not explicitly compute the numbers in Cartesian form unless it is required. Try to compute the magnitude and phase using what you know about the magnitude and phase of products of complex numbers.

   a) $(1-j)\left(\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)$

   b) $e^{j\pi/2}(1+j)(1+3j)$

   c) $je^{j\pi/3}$

   d) $e^{j\pi/4}+e^{j3\pi/4}$

   e) $(1+3j)^2$

   f) $(1-3j)/(1+3j)^2$

   g) $e^{j\pi/5}\times e^{j2\pi/5}\times e^{j3\pi/5}\ldots e^{j9\pi/5}$

   h) $e^{j\pi/5}\times e^{j2\pi/5}\times e^{j3\pi/5}\ldots e^{j9\pi/5}\times e^{j10\pi/5}$

4. Let $z_1=1, z_2=-\frac{1}{2}+j\frac{\sqrt{3}}{2}, z_3=-\frac{1}{2}-j\frac{\sqrt{3}}{2}$

   a) What are z_1^3, z_2^3 and z_3^3?

   b) Show that $z_3=z_2^2$

   c) Show that $z_1+z_2+z_3=0$

   Can you now see why z_1, z_2, z_3 can be called the cube roots of unity. They are usually expressed as $1, \omega, \omega^2$. Part c shows that the sum of the cube roots of unity is zero. In one of the homework problems, we will show that this true for nth roots of unity for any n.

5. Let $z_1 = 2e^{j\pi/4}$ and $z_2 = 8e^{j\pi/3}$. Find and express your answer in Cartesian and polar form

   a) $2z_1 - z_2$
   b) $\frac{1}{z_1}$
   c) $\frac{z_1}{z_2^2}$
   d) $\sqrt[3]{z_2}$

6. Let z be any complex number. Is it true that $(e^z)^\star = e^{z^\star}$?

7. Prove that
$$\int e^{ax} \cos(bx)\, dx = \frac{e^{ax}}{a^2 + b^2} \left(a\cos(bx) + b\sin(bx)\right)$$
   You can use integration tricks you learned in your calculus class to solve this problem. That is not the point of the exercise. Try using Euler's identity and then using integration of exponentials to see if you can solve the problem.

8. Plot the magnitude and phase of the function $X(f) = e^{j\pi f} + e^{j5\pi f}$, for $-1 \leq f \leq 1$.

9. Just for fun - what is j^j?

1.4.1 References

A good online reference for complex numbers is the wiki page `http://en.wikipedia.org/wiki/Complex_number`.

1.5 Geometric Series

A sequence of the form $a, ar, \ldots, ar^n, \ldots$, where a and r can be any *complex* number is called a geometric sequence. The first term in the sequence is a and the ratio of any two adjacent terms is r, which is called the common ratio. The partial sum S_N defined as

$$S_N := a + ar + ar^2 + \ldots + ar^{N-1} = \sum_{n=0}^{N-1} ar^n$$

is the sum of the first N terms of the sequence and is called a geometric series. Notice that the sum starts at ar^0 and goes up to ar^{N-1}. Notice that lowercase n is used as an index for the summation and uppercase N is the number of terms. It might seem confusing at first and even annoying as to why I used lower case n and upper case N to mean two different things. Throughout the course, we will have to sum signals which are indexed by time and it is common to use n to represent a time index and N is commonly used to refer to the number of terms. Therefore, it is better to get used to this notation. The sum S_N can be computed using the formula

$$\boxed{S_N = \sum_{n=0}^{N-1} ar^n = \begin{cases} \frac{a(1-r^N)}{1-r}, & r \neq 1; \\ aN, & r = 1. \end{cases}} \tag{1.21}$$

Using the same idea as before, the following general formula can be derived

$$\boxed{\text{For } N_2 > N_1, \sum_{n=N_1}^{N_2} r^n = \begin{cases} \frac{r^{N_1} - r^{N_2+1}}{1-r}, & r \neq 1; \\ N_2 - N_1 + 1, & r = 1. \end{cases}} \tag{1.22}$$

The above formula is valid for any $N_2 > N_1$ regardless of whether N_1, N_2 are positive, zero or negative. While the formula for the sum a geometric sequence is straight forward, students often have difficulty recollecting this from memory. You *must memorize* this formula !

The following special cases of (1.22) are typically encountered

$$\boxed{\sum_{n=0}^{N} r^n = \begin{cases} \frac{1-r^{N+1}}{1-r}, & r \neq 1; \\ N+1, & r = 1. \end{cases}} \tag{1.23}$$

Another special case of the above result is when $N_2 \to \infty$. In this case, the sum of infinite terms converges or diverges depending on whether $|r| < 1$ or $|r| > 1$.

$$\boxed{\sum_{n=N_1}^{\infty} r^n = \begin{cases} \frac{r^{N_1}}{1-r}, & |r| < 1; \\ \text{does not converge}, & |r| \geq 1. \end{cases}} \tag{1.24}$$

Example 1.5.1. *For example,* $1, \frac{1}{2}, \frac{1}{4}, \ldots$ *is an infinite geometric series with* $a = 1$, $b = \frac{1}{2}$.

$$S_{11} = 1 + \frac{1}{2} + \frac{1}{4} + \ldots + \ldots + \frac{1}{1024} = \frac{1 - \frac{1}{2^{11}}}{1 - \frac{1}{2}}$$

You may have seen these before, but in this class often we will be interested in the case when a and/or b are complex numbers. Luckily, nothing changes from when a and b are just real numbers.

An additional difficulty students often face is in recognizing that a given sum is actually a sum of a geometric sequence and in properly identifying a and b. The practice problems in this section as well as the homework problems should give you some practice.

The following identity is also true, although we will not use this often in this class.

$$\sum_{n=0}^{\infty} nr^n = \frac{r}{(1-r)^2}, \quad |r| < 1 \tag{1.25}$$

1.6 Practice Problems

1. Compute $5 + \frac{10}{3} + \frac{20}{9} + \frac{40}{27} + \ldots$
2. Compute $5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \ldots$
3. Simplify $\sum_{n=2}^{9} 2^{3n}3^{-2n}$
4. Compute $\sum_{n=2}^{\infty} 2^{3n}3^{-2n}$
5. Compute $\sum_{n=-\infty}^{-2} 2^{-3n}3^{2n}$. See if you can substitute $m = -n$ and obtain the expression in the previous problem
6. Simply $\sum_{n=2}^{\infty} x^n 3^{-n}$ an expression as a rational function of x. Evaluate this function for $x = 2$
7. Compute $\sum_{n=1}^{\infty} \cos^n(\pi t)$ and express as a function of t
8. Compute $\sum_{n=1}^{\infty} \cos(n\pi t)$ and express as a function of t. Hint: Use Euler's formula to convert this sum of two geometric series.
9. Simplify $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n e^{j\omega n}$
10. Compute $e^{j\frac{\pi}{2}} + \frac{1}{2}e^{j\pi} + \frac{1}{4}e^{j\frac{3\pi}{2}} + \ldots + \frac{1}{2^9}e^{j\frac{10\pi}{2}}$. Simplify your answer into a complex number in Cartesian form
11. Prove the result in (1.23) and (1.24)
12. For any two given integers k and N, what is $\sum_{n=0}^{N-1} e^{\frac{j2\pi kn}{N}}$?
13. Just for intellectual curiosity - Can you prove the results in (1.21) and (1.25)?

1.7 Integrals of complex functions and integration by Parts

In this class, we will encounter integrals of complex functions of a real variable integrated with respected to the real variable. Such integrals can always be split into two integrals - corresponding to the real and imaginary parts of the integrand and each of them will be a real function of a real variable. However, this approach will be very cumbersome in many cases as it will not be easy to split the integral into real and imaginary parts separately. It will be easier to directly integrate the complex function. All the rules of integration of real functions of real variables apply directly and j can simply be treated as a constant. In this class, we are typically interested in time t or frequency ω being the independent variable. Hence, the integrals will be with respect to t or ω.

For example,

Example 1.7.1. *Evaluate* $\int_0^{\pi/4} e^{j2t}\ dt$

$$\int_0^{\pi/4} e^{j2t}\ dt = \left[\frac{e^{j2t}}{2j}\right]_0^{\pi/4} = \frac{j-1}{2j}$$

Example 1.7.2. *Evaluate* $\int_0^{\infty} e^{(-2+j3\pi)t}\ dt$

$$\begin{aligned}\int_0^{\infty} e^{(-2+j3\pi)t}\ dt &= \left[\frac{e^{(-2+j3\pi)t}}{-2+j3}\right]_0^{\infty} \\ &= 0 - \frac{1}{-2+j3} = \frac{1}{2-j3}\end{aligned}$$

Sometimes we will have to use Integration by parts to evaluate integrals. The main result to recall is

$$\int_a^b u(t)\ dv(t)\ dt = [u(t)\ v(t)]_a^b - \int_a^b v(t)\ du(t)\ dt \tag{1.26}$$

Example 1.7.3. *Evaluate* $\int_0^1 te^{-j\omega t}dt$*. Your answer should clearly be a function of* ω

We choose

$$\begin{aligned}u(t) = t \quad &, \quad dv(t) = e^{-j\omega t}\ dt \\ \Longrightarrow du(t) = dt \quad &, \quad v(t) = \frac{e^{-j\omega t}}{-j\omega}\end{aligned}$$

$$\begin{aligned}\int_0^1 te^{-j\omega t}dt &= \left[\frac{te^{-j\omega t}}{-j\omega}\right]_0^1 - \int_0^1 \frac{e^{-j\omega t}}{-j\omega}\ dt \\ &= -e^{-j\omega} - \left[\frac{e^{-j\omega t}}{-\omega^2}\right]_0^1 = -e^{-j\omega} + \frac{e^{-j\omega}-1}{\omega^2}\end{aligned}$$