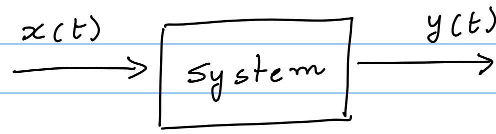
Memoryless Systems and Systems with memory

Note Title 6/26/2011



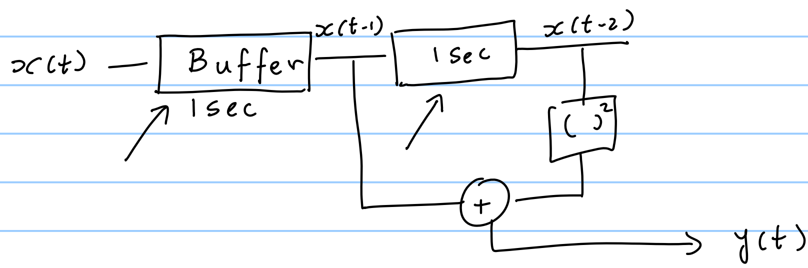
A system is said to have memory if $y(t_0)$ depends on any $x(t_1)$ where $t_1 \neq t_0$

Example of a system that has memory

1)

$$y(t) = x(t-1) + \left(x(t-2)\right)^2$$

$$y(t_0) = x(t_0-1) + \left(x(t_0-2)\right)^2$$



2)

$$y(t) = 2\,x(t+1) - 3\,x(t)$$

Examples of Memoryless Systems

* $$y(t) = \left(x(t)\right)^3$$

* $$y(t) = \cos(t+1)\;x(t)$$

$$y(t) = \sin(t-2)\cdot x(t)$$

do not get confused by this

Example

$$y[n] = y[n-1] + x[n]$$

$y[0] = 0 \quad x[0] = 0$

$y[1] = x[1]$

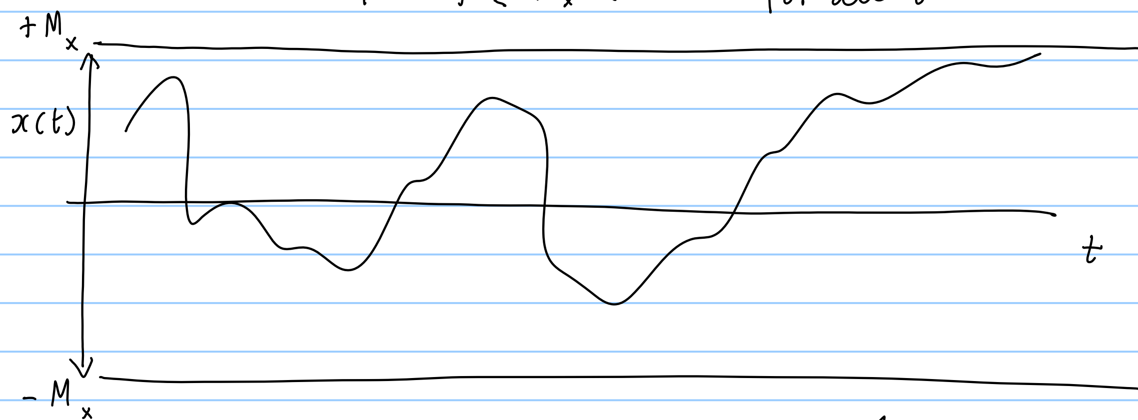
$\Longrightarrow y[2] = y[1] + x[2]$

$= x[1] + x[2]$

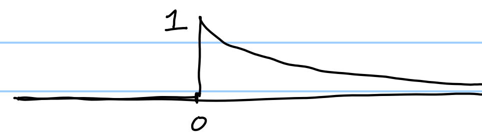
Bounded Input Bounded Output (BIBO) stability

A signal $x(t)$ is said to be bounded if

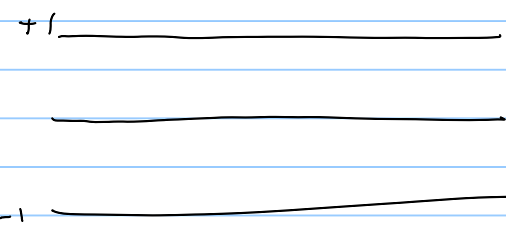
$$|x(t)| \le M_x < \infty \qquad \text{for all } t$$



Examples $\quad x(t) = e^{-t}\,u(t)$



$|x(t)| \le 1$ for all t $\quad M_x = 1$

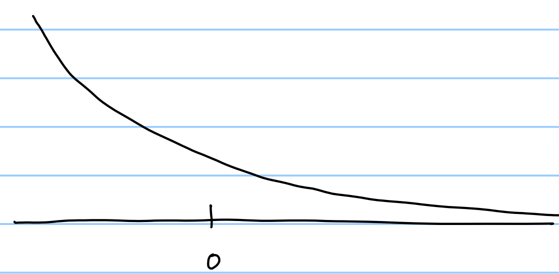


$x(t) = \cos(t)$

$M_x = 1$

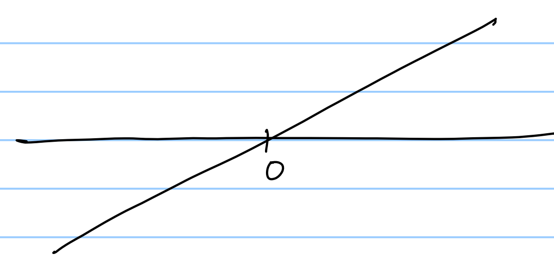
Unbounded Signals

$x(t) = e^{-t}$



unbounded

$x(t) = t$



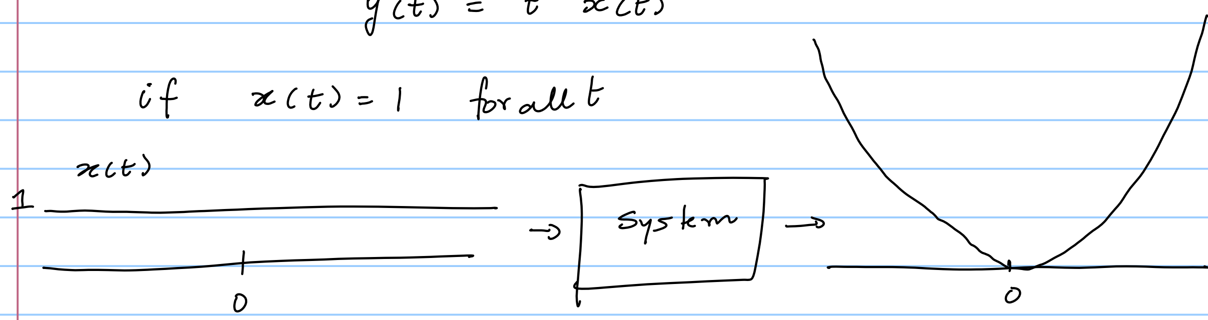
BIBO Stability

A system is stable (BIBO) if every bounded input (signal) produces a bounded output (signal)

Examples

$$y(t) = t^2 x(t)$$

if $x(t) = 1$ for all t



bounded input — unbounded M_y

NOT STABLE

* $y(t) = e^{x(t)}$

if $|x(t)| \le M_x$ then $|y(t)| \le \underbrace{e^{M_x}}_{M_y}$

STABLE

$y(t) = e^t x(t)$ ← UNSTABLE

Linearity

$$\text{input}(t) \longrightarrow \boxed{H} \longrightarrow \text{output}(t)$$

$$x_1(t) \xrightarrow{H} y_1(t)$$

* Additivity or Superposition

$$x_1(t) \longrightarrow y_1(t)$$
$$x_2(t) \longrightarrow y_2(t)$$

$$x_3(t) = x_1(t) + x_2(t) \longrightarrow y_3(t) = y_1(t) + y_2(t)$$

Examples

$$y(t) = t\, x(t) \qquad \underbrace{\text{output}(t) = t \ \text{input}(t)}$$

$$x_1(t) \longrightarrow y_1(t) = t\, x_1(t)$$
$$x_2(t) \longrightarrow y_2(t) = t\, x_2(t)$$

$$x_3(t) = x_1(t) + x_2(t) \longrightarrow y_3(t) = t\, x_3(t)$$
$$= t \cdot (x_1(t) + x_2(t))$$
$$= \underbrace{t \cdot x_1(t)}_{y_1(t)} + \underbrace{t \cdot x_2(t)}_{y_2(t)}$$

Satisfies Additive property

Example

$$y(t) = x^2(t)$$
$$y(t) = x(t)\, x(t-1)$$

$$\text{output}(t) = \text{input}(t)\ \text{input}(t-1)$$

$$x_1(t) \longrightarrow y_1(t) = x_1(t)\, x_1(t-1)$$
$$x_2(t) \longrightarrow y_2(t) = x_2(t)\, x_2(t-1)$$
$$x_3(t) = x_1(t) + x_2(t) \longrightarrow y_3(t) = x_3(t)\, x_3(t-1)$$
$$= \left[(x_1(t) + x_2(t))(x_1(t-1) + x_2(t-1))\right]$$

$$= \underbrace{x_1(t)\, x_1(t-1)}_{y_1(t)} + \underbrace{x_2(t)\, x_2(t-1)}_{y_2(t)} + \underbrace{x_1(t) x_2(t-1) + x_2(t)\, x_1(t-1)}$$

$$y_3(t) \neq y_1(t) + y_2(t)$$

Homogenity

If $x_1(t) \longrightarrow y_1(t)$

and $a\,x_1(t) \longrightarrow a\,y_1(t)$ for all a and $x_1(t)$

Examples: $y(t) = t\,x(t)$ is homogenous

$y(t) = [x(t)]^2$ is not homogenous

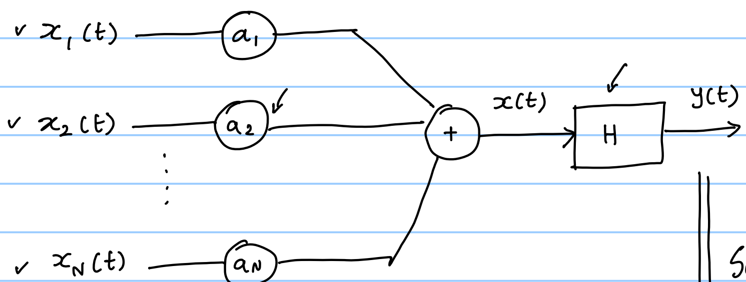
A System is Linear if it is both Additive and Homogenous

$$a_1 x_1(t) \longrightarrow a_1 y_1(t)$$
$$a_2 x_2(t) \longrightarrow a_2 y_2(t)$$
$$\vdots$$
$$a_N x_N(t) \longrightarrow a_N y_N(t)$$

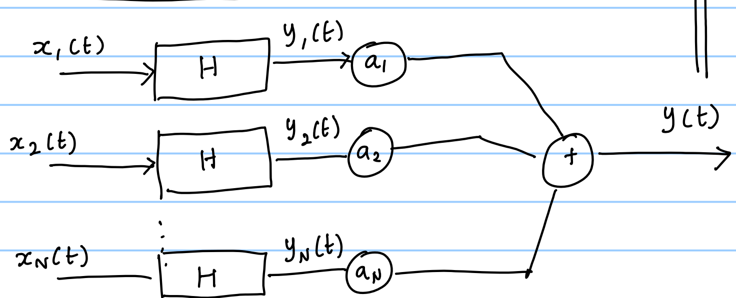
$$\boxed{x(t) = \sum_{i=1}^{N} a_i x_i(t) \longrightarrow y(t) = \sum_{i=1}^{N} a_i y_i(t)}$$

If $x(t) = \sum a_i x_i(t)$ then

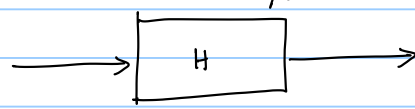
$$\boxed{y(t) = H\{x(t)\} = H\left\{\sum a_i x_i(t)\right\} = \sum a_i H\{x_i(t)\}}$$



Same!



Example Consider a Linear System H



$$\left.\begin{aligned} x_1(t) &= e^{j2\pi t} \longrightarrow y_1(t) = e^{j3\pi t} \\ x_2(t) &= e^{-j2\pi t} \longrightarrow y_2(t) = e^{-j3\pi t} \end{aligned}\right\}$$ we are given this

$$x_3(t) = \sin(2\pi t) \longrightarrow ?$$

$$\sin 2\pi t = \underset{\substack{\downarrow \\ a_1}}{\frac{1}{2j}} \underbrace{e^{j2\pi t}}_{x_1(t)} + \underset{\substack{\downarrow \\ a_2}}{\left(-\frac{1}{2j}\right)} \underbrace{e^{-j2\pi t}}_{x_2(t)} \longrightarrow \frac{1}{2j} e^{j3\pi t} - \frac{1}{2j} e^{-j3\pi t}$$

$$= \sin(3\pi t)$$

$$x_4(t) = \sin\left(2\pi\left(t - \frac{1}{\pi}\right)\right) \longrightarrow ?$$

Time Invariance

$$x_1(t) \longrightarrow \boxed{H} \longrightarrow y_1(t) = H\{x_1(t)\}$$

Parameters do not change with time   Time Invariant

$$y_1(t) = K\, x_1(t) \quad \leftarrow \text{Time Invariant}$$

$$y_1(t) = K(t)\, x_1(t)$$
$$= \frac{1}{t}\, x_1(t)$$

If the output corresponding to a delayed version of $x_1(t)$ is the output corresponding to $x_1(t)$ delayed by the same amount.

$$x_1(t) \longrightarrow \boxed{S^{t_0}} \longrightarrow x_2(t) = x_1(t-t_0)$$

$$x_1(t) \longrightarrow \boxed{S^{t_0}} \xrightarrow{x_2(t) = x_1(t-t_0)} \boxed{H} \longrightarrow y_2(t)$$

$$x_1(t) \longrightarrow \boxed{H} \xrightarrow{y_1(t) = H\{x_1(t)\}} \boxed{S^{t_0}} \longrightarrow y_1(t-t_0)$$

$y_2(t) = y_1(t-t_0)$ = Time Invariant

$$\boxed{H S^{t_0}\{x_1(t)\} = S^{t_0} H\{x_1(t)\}}$$ Time Invariant

Example 1:

$$y(t) = \cos(x(t))$$

$$x_1(t) \longrightarrow \boxed{S^{t_0}} \xrightarrow{x_2(t) = x_1(t-t_0)} \boxed{H} \longrightarrow y_2(t) = \cos(x_2(t))$$

$$y_2(t) = \cos(x_1(t-t_0))$$

$$\parallel$$

$$x_1(t) \longrightarrow \boxed{H} \xrightarrow{y_1(t) = \cos(x_1(t))} \boxed{S^{t_0}} \longrightarrow y_1(t-t_0) = \cos(x_1(t-t_0))$$

replace t by $t-t_0$ in $y_1(t)$

Example

$$y(t) = x(2t)$$

$$x_1(t) \longrightarrow \boxed{S^{t_0}} \xrightarrow{x_2(t) = x_1(t-t_0)} \boxed{H} \longrightarrow y_2(t) = x_2(2t) \quad \checkmark$$

$$y_2(t) = x_1(2t - t_0)$$

$$\neq$$

$$x_1(t) \longrightarrow \boxed{H} \xrightarrow[t-t_0]{y_1(t) = x_1(2t)} \boxed{S^{t_0}} \longrightarrow y_1(t-t_0) = x_1(2(t-t_0))$$

$$= x_1(2t - 2t_0) \quad \checkmark$$

Time variant !

Example

Problem 1.27 d) from Oppenheim, Wilsky and Young

Consider the system described by

$$y(t) = \begin{cases} 0 & t < 0 \\ x(t) + x(t-2) & t \geq 0 \end{cases}$$

Is this system Time Invariant ?

$$x_1(t) \longrightarrow \boxed{S^{t_0}} \xrightarrow{x_2(t) = x_1(t-t_0)} \boxed{H} \longrightarrow y_2(t) = \begin{cases} 0 & t < 0 \\ x_1(t-t_0) + x_1(t-2-t_0) & t \geq 0 \end{cases}$$

$$x_1(t) \longrightarrow \boxed{H} \xrightarrow{y_1(t) = \begin{cases} 0 & t<0 \\ x_1(t) + x_1(t-2) & t \geq 0 \end{cases}} \boxed{S^{t_0}} \longrightarrow y_1(t-t_0) = \begin{cases} 0 & t - t_0 < 0 \\ x_1(t-t_0) + x_1(t-t_0-2) & t - t_0 \geq 0 \end{cases}$$

Notice the difference in the intervals over which the signals $y_2(t)$ and $y_1(t-t_0)$ are non-zero. Hence the system is <u>NOT time-invariant</u>

Construct a simple counter example now!

Causality

$$x[n] \longrightarrow \boxed{H} \longrightarrow y[n]$$

System is non-causal (or anticipative) if the output at time n_0 ($y[n_0]$) depends on any future value of the input, i.e ($x[k]$ for any $k > n_0$)

System 1 : $y[n] = \frac{1}{3}\left(x[n] + x[n-1] + x[n-2] \right)$ ✓ Causal

System 2 : $y[n] = \frac{1}{3}\left(x[n-1] + x[n] + x[n+1] \right)$ Non-causal

Examples $y[n] = x[-n]$

For negative values of n

$y[-1] = x[1]$ NOT CAUSAL

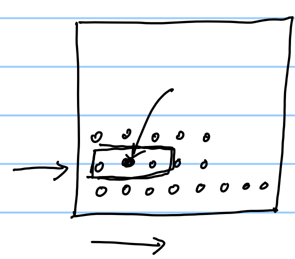
$$y(t) = \sin(t+1)\, x(t)$$

CAUSAL

Why do we care about non-causal systems ?

Two examples

Example 1 : when the independent variable is not time

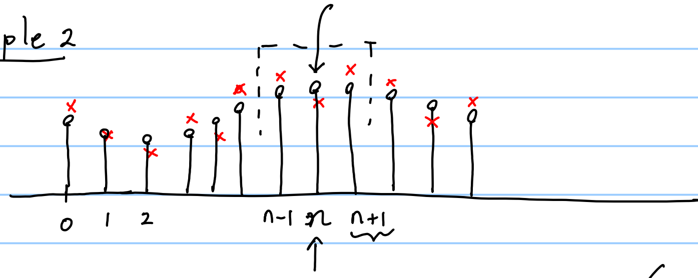


$x[n]$

$y[n] = \frac{1}{3}\left(x[n-1] + x[n] + x[n+1] \right)$

Non-causal

Example 2



$x[n] \leftarrow$ noisy signal

$$y[n] = \frac{1}{3}\left(x[n-1] + x[n] + x[n+1]\right)$$

$$\xrightarrow{x[n]} \quad \begin{bmatrix} x[n+1] & x[n] & x[n-1] \end{bmatrix}$$

$$\tilde{x}[n] = x[n+1]$$

$$y[n] = \frac{1}{3}\left(\tilde{x}[n-2] + \tilde{x}[n-1] + \tilde{x}[n]\right)$$

Non-causal system can be converted to causal systems with an appropriate delay.