Note Title 6/23/2012

# Determining Properties of LTI Systems based on the impulse response

Given the impulse response $h[n]$ or $h(t)$ of an LTI system, can we say if the system is memoryless, causal, stable and invertible?

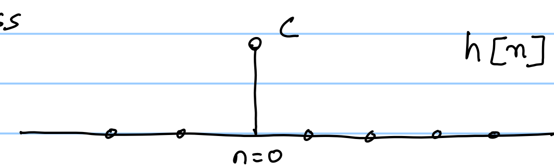
**Memoryless**:

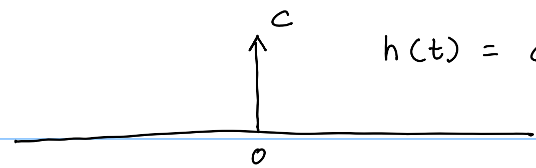
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] = \boxed{\sum_{k=-\infty}^{\infty} h[k]\, x[n-k]}$$

$y[n]$ should depend only on the value of the input at the same time. $y[n]$ cannot depend on $x[n-k]$ for any $k \neq 0$

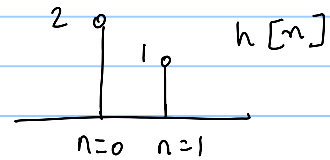
$h[k] = 0$ for all $k \neq 0$

If $h[k] = 0$ for all $k \neq 0$ or $h[n] = 0$ for all $n \neq 0$ then the LTI system is memoryless





$h(t) = c\,\delta(t)$

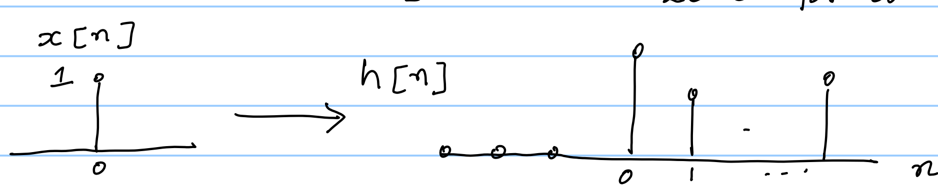
**Example:** $h[n] =$  $y[n] = 2x[n] + x[n-1]$

**Causality:**

$$y[n] = \boxed{\sum_{k=-\infty}^{\infty} h[k]\, x[n-k]}$$

Namely $y[n]$ does not depend on $x[n+k]$ for any $k>0$
$x[n-k]$ for any $k<0$

$h[k]$ must be zero for all $k<0$



How to determine if an LTI System is stable from the impulse response ?

Impulse response - $h[n]$ or $h(t)$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

We want $y[n]$ to be bounded if $x[n]$ is bounded.

$$|y[n]| = \left| \sum h[k]\, x[n-k] \right| \leq \sum \left| h[k]\, x[n-k] \right|$$

$$= \sum_{k=-\infty}^{\infty} |h[k]|\, |x[n-k]|$$

Since $x[n]$ is bounded, $|x[n-k]| < M_x$

$$\boxed{|y[n]| \leq M_x \left( \sum_{k=-\infty}^{\infty} |h[k]| \right)}$$

$\boxed{\text{If } \sum_{n=-\infty}^{\infty} |h[n]| < M_h \text{ (is bounded)}}$ then the LTI system is stable

The LTI system is stable if and only if $\sum_{n=-\infty}^{\infty} |h[n]| < M_h$ (is bounded)

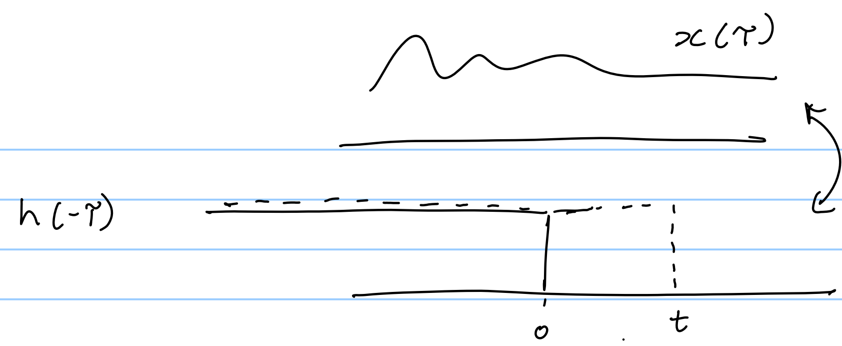
$\int_{-\infty}^{\infty} |h(t)| < M_h$ (is bounded)

**Example:** $\boxed{h(t) = u(t)}$

$\int_{-\infty}^{\infty} |h(t)|\, dt = \int_{0}^{\infty} 1 \cdot dt$ is not bounded

UNSTABLE

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$




$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$

**Example 2**

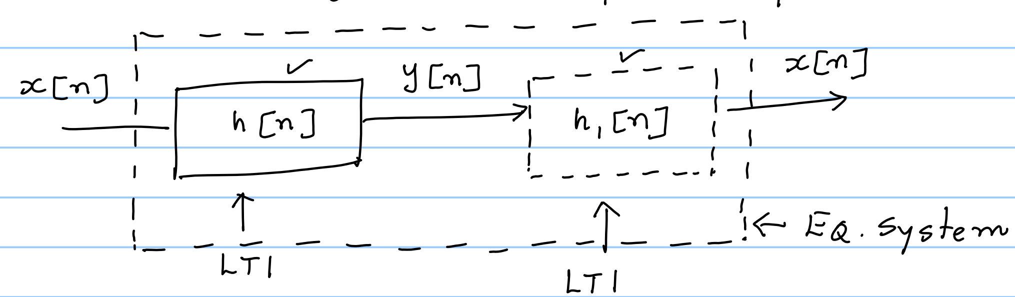
$$h[n] = \left(\frac{1}{2}\right)^n \underbrace{u[n]}$$

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = 2 \quad \text{bounded}$$

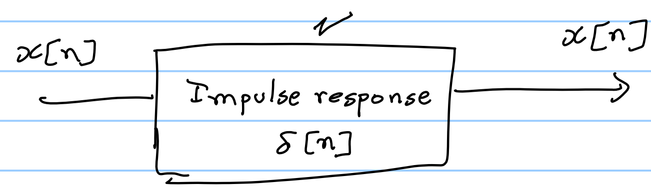
so the system is stable

When is an LTI system invertible?

Suppose $h[n]$ or $h(t)$ is the impulse response




If an LTI system is invertible it has an LTI inverse




$$\boxed{\begin{aligned} h[n] * h_1[n] &= \delta[n] \\ h(t) * h_1(t) &= \delta(t) \end{aligned}}$$

The LTI system with impulse response $h[n]$ or $h(t)$ is invertible if there exists an $h_1[n]$ or $h_1(t)$ s.t

## Examples

$h(t) = \delta(t - t_0)$

$$y(t) = x(t) * \delta(t - t_0) = x(t - t_0)$$

$$x(t) \longrightarrow \boxed{\begin{matrix}\text{shift by}\\ t_0\end{matrix}} \xrightarrow{\;y(t) = x(t - t_0)\;} \boxed{\begin{matrix}\text{shift by}\\ -t_0\end{matrix}} \longrightarrow x(t)$$

LTI      LTI

$h(t) = \delta(t - t_0)$      $h_1(t) = \delta(t + t_0)$

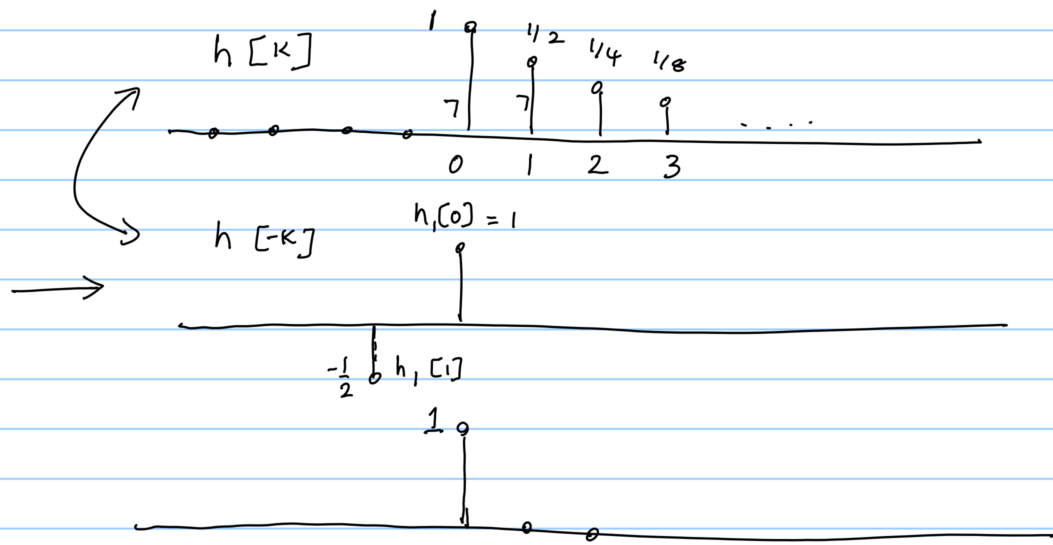
$$\delta(t - t_0) * \delta(t + t_0) = \delta(t)$$

$$\delta(t - t_0) * \delta(t) = \delta(t - t_0)$$

$$\delta(t - t_0) * \delta(t + t_0) = \delta(t + t_0 - t_0) = \delta(t)$$

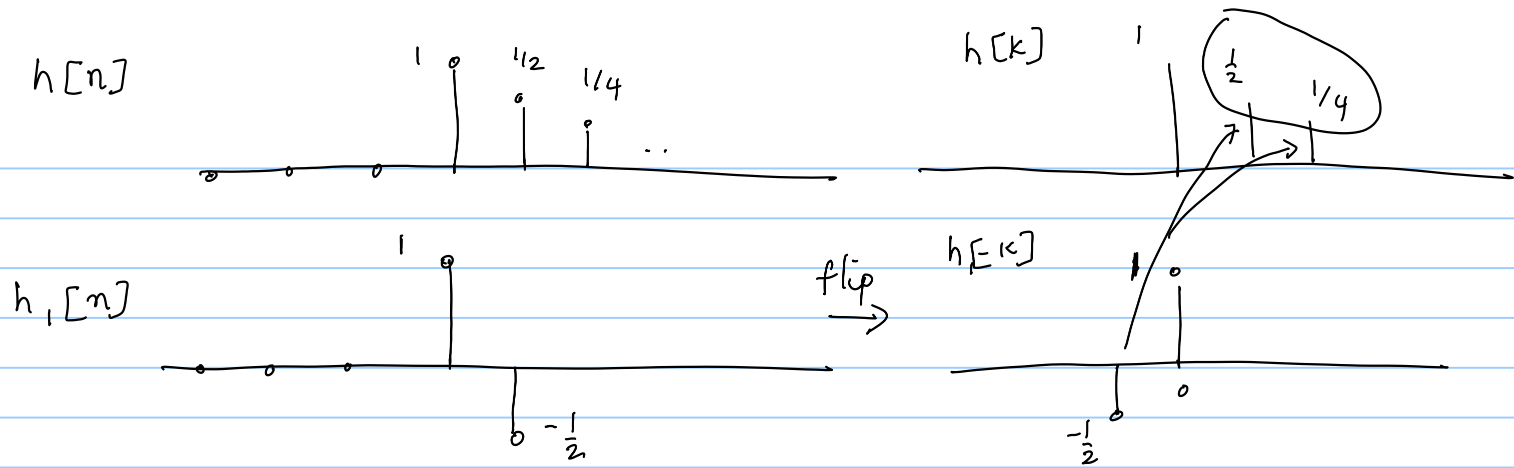
**Example** $h[n] = \left(\frac{1}{2}\right)^n u[n]$

Is there an $h_1[n]$ such that $h[n] * h_1[n] = \delta[n]$




$h_{eq}[n] = h[n] * h_1[n]$

$h_{eq}[2] = 1 \cdot \frac{1}{4} - \frac{1}{2} \cdot \frac{1}{2} + h_1[2] \cdot 1 = 0 \Rightarrow h_1[2] = 0$




$h_1[n] = \delta[n] - \frac{1}{2}\delta[n-1]$ is the inverse of $h[n] = \left(\frac{1}{2}\right)^n u[n]$