FT of Periodic Signals

So far we have studied

Fourier Series for periodic signals
Fourier Transform for Non-periodic signals

By defining the FT of Periodic signals, we can use only the FT to analyze all signals under a common framework.

Begin by recalling

$$\delta(t) \longleftrightarrow 1$$
$$1 \longleftrightarrow 2\pi\,\delta(\omega)$$
$$e^{j\omega_0 t} \longleftrightarrow 2\pi\,\delta(\omega-\omega_0)$$
$$\delta(t-T) \longleftrightarrow e^{-j\omega T}$$

Say $x(t)$ is periodic and let $X[k]$ be the F.S. coefficients.

$$x(t) = \sum_{k=-\infty}^{\infty} X[k]\, e^{jk\omega_0 t}$$

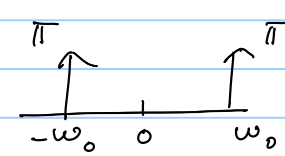
$$\therefore \quad \boxed{X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} X[k]\, \delta(\omega - k\omega_0)}$$

Examples

$\ast \quad x(t) = \cos\omega_0 t = \frac{1}{2} e^{j\omega_0 t} + \frac{1}{2} e^{-j\omega_0 t}$

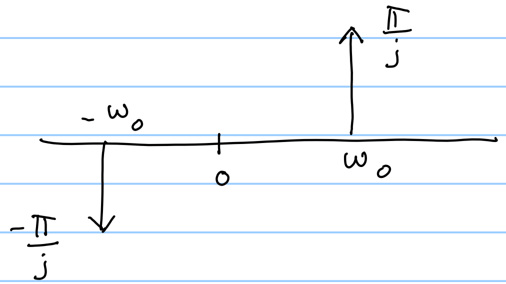
$X[1] = \frac{1}{2} \quad X[-1] = \frac{1}{2}$

$X(j\omega) = \pi\,\delta(\omega-\omega_0) + \pi\,\delta(\omega+\omega_0)$

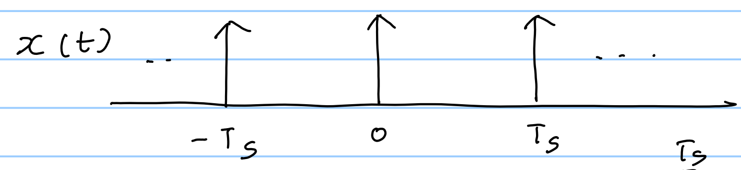


* $x(t) = \sin\omega_0 t = \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t}$

$$X(j\omega) = \frac{\pi}{j}\,\delta(\omega-\omega_0) - \frac{\pi}{j}\,\delta(\omega+\omega_0)$$



* $x(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT_s)$



Time period $= T_s$ $\qquad X[k] = \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} \delta(t)\, e^{-j\omega t} = \frac{1}{T_s}$

$\omega_0 = \frac{2\pi}{T_s}$

$$\boxed{\begin{aligned} X(j\omega) &= \frac{2\pi}{T_s}\sum_{k=-\infty}^{\infty}\delta(\omega-k\omega_0) \\ &= \frac{2\pi}{T_s}\sum_{k=-\infty}^{\infty}\delta\left(\omega-\frac{2\pi}{T_s}\,\omega_0\right) \end{aligned}}$$

* Parseval's Relation

$$\frac{1}{T}\int_{-\infty}^{\infty}|x(t)|^2\,dt = \left[\frac{1}{2\pi}\int_{-\infty}^{\infty}|X(j\omega)|^2\,dt\right]\frac{1}{T}$$

Suppose $x(t)$ is periodic

$$X(j\omega) = 2\pi \cdot \sum_{k} \delta(\omega - k\omega_0)\; X[k]$$

$$\boxed{\text{Finally, } \quad \frac{1}{T} \int_{T} |x(t)|^2 \, dt = \sum_{k=-\infty}^{\infty} |X[k]|^2}$$