Note Title 7/12/2012

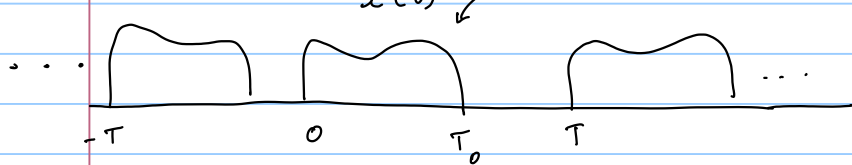
Continuous-time Fourier transform

Recall Fourier Series representation

If $\tilde{x}(t)$ is periodic with timeperiod T

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} \tilde{X}[k]\, e^{jk\omega_0 t} \longleftarrow$$

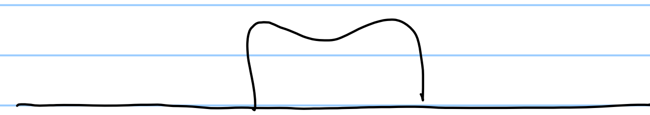
$$\tilde{X}[k] = \frac{1}{T} \int_T \tilde{x}(t)\, e^{-jk\omega_0 t}\, dt \longleftarrow$$



$$\omega_0 = \frac{2\pi}{T} \longleftarrow$$

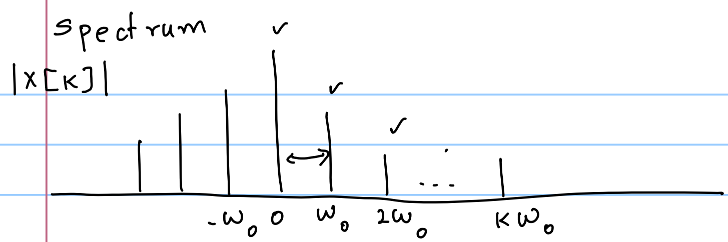
Suppose $x(t)$ is not (necessarily) periodic.

Is there a representation for $x(t)$ as a linear combination of complex exponentials ?

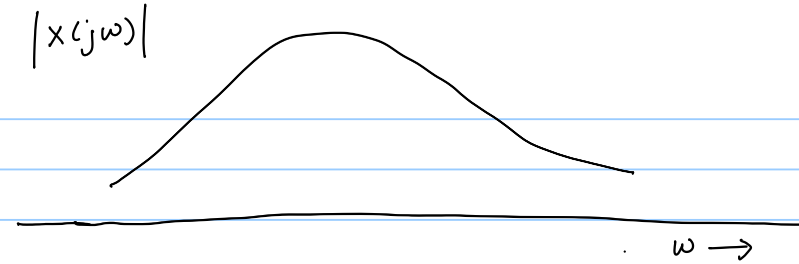


Main idea: Think of $x(t)$ as the limit of $\tilde{x}(t)$ when $T \to \infty$

$$x(t) = \lim_{T \to \infty} \tilde{x}(t)$$



Fourier transform of $x(t)$



Summary

* F.S. representation applies to periodic Signals
  A signal contains only frequencies which are integer multiples of a fundamental frequency

* F.T. representation applies to Non-periodic (and periodic) Signals
  The signal may contain a continuum of frequencies
  $X(j\omega)$ refers to the F.T, where ω is a continuosly changing Variable

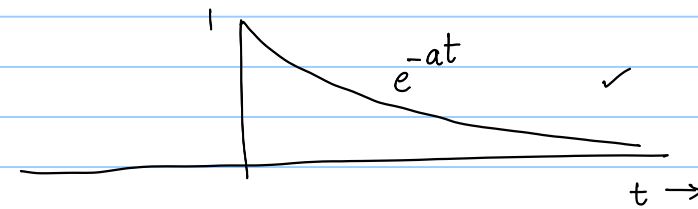
$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \quad \leftarrow \text{Analysis Equation}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega \quad \leftarrow \text{Synthesis Equation}$$

Computing the Fourier Transform

Examples

1) $x(t) = e^{-at} u(t) \quad a>0$

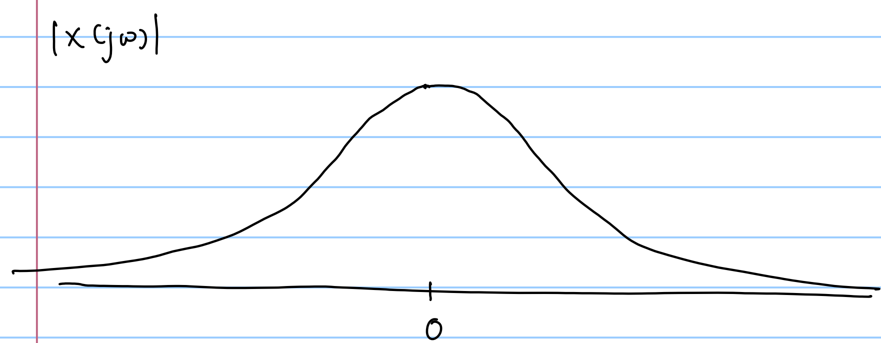


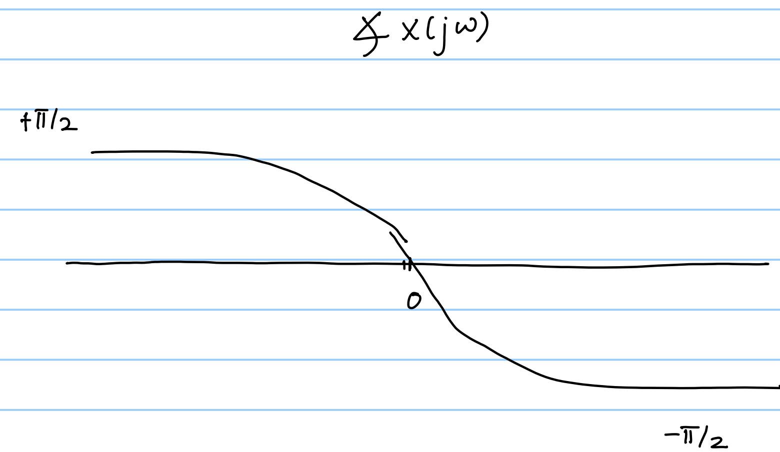
$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-at} u(t)\, e^{-j\omega t}\, dt = \int_{0}^{\infty} e^{-at} e^{-j\omega t}\, dt = \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt$$

$$X(j\omega) = \left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right]_0^{\infty} = \frac{0-1}{-(a+j\omega)} = \boxed{\frac{1}{a+j\omega}}$$

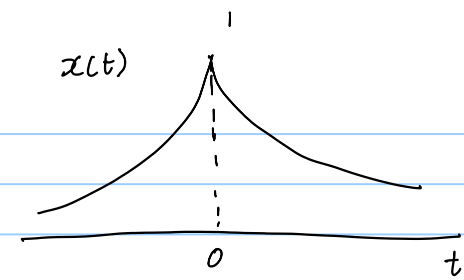
$$|X(j\omega)| = \left|\frac{1}{a+j\omega}\right| = \frac{1}{|a+j\omega|} = \boxed{\frac{1}{\sqrt{a^2+\omega^2}}}, \quad \measuredangle X(j\omega) = 0 - \tan^{-1}\left(\frac{\omega}{a}\right)$$





$$X(j\omega) = \frac{1}{a+j\omega} = \frac{a-j\omega}{a^2+\omega^2}$$

Example 2 : $x(t) = e^{-a|t|}, \quad a > 0$



$$\left.\begin{aligned} x(t) &= e^{-at} && t > 0 \\ &= e^{at} && t < 0 \end{aligned}\right\}$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} e^{at} e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} e^{(a-j\omega)t}\, dt + \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt$$

$$= \left[\frac{e^{(a-j\omega)t}}{a-j\omega} \right]_{-\infty}^{0} + \left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)} \right]_{0}^{\infty}$$

$$\boxed{X(j\omega) = \frac{1-0}{a-j\omega} + \frac{0-1}{-a+j\omega} = \frac{1}{a-j\omega} + \frac{1}{a+j\omega} = \frac{2a}{a^2+\omega^2}}$$

Example 3 : $x(t) = \delta(t)$

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t) \underbrace{e^{-j\omega t}}_{g(t)} dt = e^{-j\omega t}\Big|_{t=0} = 1$$

Recall $\displaystyle\int_{-\infty}^{\infty} \delta(t)\, g(t)\, dt = g(0) = g(t)\Big|_{t=0}$

$\boxed{X(j\omega) = 1}$ $\qquad \delta(t) \longleftrightarrow 1$