Note Title 6/30/2012

Continuous-Time Fourier series

Computing the F.S. Coefficients

$x(t)$ is a periodic signal with a time period T

Fundamental frequency $\omega_0 = \frac{2\pi}{T}$

$$x(t) = \sum_{k=-\infty}^{\infty} X[k] \, e^{jk\omega_0 t}$$

* Direct method
* Method of Inspection

Direct Method

$T \leftarrow$ Time period of $x(t)$ and $\omega_0 = \frac{2\pi}{T}$ $\boxed{\omega_0 T = 2\pi}$

$$\boxed{x(t) = \sum_{K=-\infty}^{\infty} X[K] \, e^{\underbrace{jK\omega_0 t}}}$$

$$\left\{ \ldots\ldots, e^{-j2\omega_0 t}, e^{-j\omega_0 t}, 1, e^{j\omega_0 t}, e^{j2\omega_0 t}, \ldots\ldots \right\}$$

$e^{jm\omega_0 t}$ $e^{jl\omega_0 t}$

$$\int_0^T e^{jm\omega_0 t} e^{-jl\omega_0 t} \, dt = \int_0^T e^{j(m-l)\omega_0 t} \, dt$$

$$= \left[\frac{e^{j(m-l)\omega_0 t}}{j(m-l)\omega_0} \right]_0^T = \frac{e^{j(m-l)\omega_0 T} - 1}{j(m-l)\omega_0}$$

$\omega_0 T = 2\pi$

Remember m and l are integers

$$e^{j(m-l)\omega_0 T} = e^{j(m-l)2\pi} = 1$$

$$\int_0^T e^{jm\omega_0 t} e^{-jl\omega_0 t} t = 0 \qquad m \neq l$$

$$= T \qquad m = l$$

$$\int_0^T x(t) e^{-jl\omega_0 t} dt = \int_0^T \sum X[k] e^{jk\omega_0 t} \cdot e^{-jl\omega_0 t} dt$$

$$= \sum_{k=-\infty}^{\infty} \int_0^T X[k]\, e^{j(k-l)\omega_0 t}\, dt$$

$$\int_0^T x(t)\, e^{-jl\omega_0 t}\, dt = \sum_{k=-\infty}^{\infty} X[k] \underbrace{\int_0^T e^{j(k-l)\omega_0 t}\, dt}$$

$= 0$ if $k \neq l$

$= T$ if $k = l$

$$\int_0^T x(t)\, e^{-jl\omega_0 t}\, dt = X[l] \cdot T$$

$$\boxed{X[k] = \frac{1}{T} \int_0^T x(t)\, e^{-jk\omega_0 t}\, dt}$$ ← Analysis eqn

Computing FS Coefficients by Inspection

Example: $x(t) = 3\cos\left(\frac{\pi t}{2} + \frac{\pi}{4}\right) = \sum_{k=-\infty}^{\infty} x[k]\, e^{jk\omega_0 t}$

$$3\cos\left(\frac{\pi t}{2} + \frac{\pi}{4}\right) = \cdots + x[-2]\,e^{-j2\omega_0 t} + x[-1]\,e^{-j\omega_0 t} + x[0] + x[1]\,e^{j\omega_0 t} + x[2]\,e^{j2\omega_0 t} + \cdots$$

$$\frac{3}{2}\left[e^{j\left(\frac{\pi t}{2} + \frac{\pi}{4}\right)} + e^{-j\left(\frac{\pi t}{2} + \frac{\pi}{4}\right)}\right] =$$

Time period $T = 4$ $\quad \omega_0 = \frac{2\pi}{4} = \frac{\pi}{2}$

$$\underbrace{\frac{3}{2}e^{-j\frac{\pi}{4}}}_{x[-1]} \cdot e^{-j\frac{\pi t}{2}} + \underbrace{\frac{3}{2}e^{j\frac{\pi}{4}}}_{x[1]} \cdot e^{j\frac{\pi t}{2}} = \cdots + x[-1]\,\underbrace{e^{-j\frac{\pi}{2}t}} + x[0] + x[1]\,\underbrace{e^{j\frac{\pi t}{2}}} + x[2]\,e^{j\pi t} + \cdots$$

F.S. Coefficients are given

$$X[K] = \begin{cases} \frac{3}{2} e^{-j\pi/4} & K = -1 \\ \frac{3}{2} e^{j\pi/4} & K = 1 \\ 0 & K \neq -1, 1 \end{cases}$$

$$\boxed{X[K] = \frac{3}{2} e^{\frac{-j\pi}{4}} \delta[K+1] + \frac{3}{2} e^{j\pi/4} \delta[K-1]}$$

$$x(t) = \sum_{K=-\infty}^{\infty} X[K]\, e^{jK\omega_0 t}$$

fundamental frequency